AD-A276 471

CMS Technical Summary Report #94-6

STRICTLY POSITIVE DEFINITE FUNCTIONS ON SPHERES

Amos Ron and Xingping Sun

UNIVERSITY OF WISCONSIN

CENTER FOR THE MATHEMATICAL SCIENCES

Center for the Mathematical Sciences
University of Wisconsin—Madison
1308 W. Dayton Street
Madison, Wisconsin 53715-1149

February 1994

(February 7, 1994)

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
1800 G Street
Washington, DC 20550

UNIVERSITY OF WISCONSIN-MADISON
CENTER FOR THE MATHEMATICAL SCIENCES

Strictly Positive Definite Functions on Spheres

Amos Ron[1,*] & Xingping Sun[2]

Technical Summary Report # 94-6
February 1994

ABSTRACT

In this paper we study strictly positive definite functions on the unit sphere of the m-dimensional Euclidean space. Such functions can be used for solving a scattered data interpolation problem on spheres. Since positive definite functions on the sphere were already characterized by Schoenberg some fifty years ago, the issue here is to determine what kind of positive definite functions are actually strictly positive definite. The study of this problem was initiated recently by Xu and Cheney, [XC], where certain sufficient conditions were derived. A new approach, which is based on a critical connection between this problem and that of multivariate polynomial interpolation on spheres, is presented here. The relevant interpolation problem is subsequently analysed by three different complementary methods. The first is based on the de Boor-Ron general "least solution for the multivariate polynomial interpolation problem". The second, which is suitable only for $m = 2$, is based on the connection between bivariate harmonic polynomials and univariate analytic polynomials, and reduces the problem to the structure of the integer zeros of bounded univariate exponentials. Finally, the last method invokes the realization of harmonic polynomials as the polynomial kernel of the laplacian, thereby exploiting some basic relations between homogeneous ideals and their polynomial kernels.

1991 Mathematics Subject Classification: Primary 42A82, 41A05: secondary 33C55, 33C90

Authors' affiliation and address:

[1] Computer Science Department
1210 West Dayton St.
University of Wisconsin-Madison
Madison, WI 53706 USA
amos@cs.wisc.edu

and

[2] Department of Mathematics
Southwest Missouri State University
Springfield, MO 65804
USA
xis280f@CSM560.smsu.edu

* supported by the National Science Foundation under Grants DMS-9102857, DMS-9224748, and by the United States Army under Contract DAAL03-G-90-0090,

Strictly Positive Definite Functions on Spheres

Amos Ron & Xingping Sun

1. Introduction

Let S^{m-1} denote the unit sphere in the Euclidean space $\mathbb{R}^m$ $(m \geq 2)$, and d_m the geodesic distance on S^{m-1}, i.e.,

$$d_m(x,y) = \mathrm{Arccos}(x \cdot y), \quad x, y \in S^{m-1}.$$

Here $x \cdot y$ denotes the usual inner product of x and y. Let $g : [0,\pi] \to \mathbb{R}$ be a continuous function, and let $\Theta = \{\theta_1, \ldots, \theta_n\}$ be a subset of S^{m-1}. We study in this paper the possible strict positive definiteness of the $n \times n$ matrix

$$A := A_\Theta := A_{g,\Theta} \tag{1.1}$$

with ij–entry

$$A_{ij} = g(d_m(\theta_i, \theta_j)); \tag{1.2}$$

i.e., we look for conditions such that

$$c^T A c = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j g(d_m(\theta_i,\theta_j)) > 0, \quad \forall c = (c_1,\ldots,c_n) \in \mathbb{R}^n \backslash 0.$$

The matrix A of (1.1) naturally arises in the study of approximations to scattered data on spheres. Let $\Theta \subset S^{m-1}$ be a finite set, and let f be some function defined on Θ. To approximate f by a function defined on the entire sphere, one may choose a univariate function $g : [0,\pi]$, and look for an interpolant g_f in the linear space

$$G_\Theta := \mathrm{span}\{g(d_m(\cdot,\theta)) : \ \theta \in \Theta\}.$$

The existence of a unique interpolant $g_f \in G_\Theta$ for f then amounts to the invertibility of the above matrix A. Of course, if A is also positive definite, then the finding of g_f. i.e., the inversion of A, can then be approached by efficient and stable numerical (iterative or direct) methods.

In view of the above, the following problem becomes self-suggestive:

Problem 1.3. *Determine conditions under which the interpolation matrix A_Θ of (1.1) is*

(a) Positive definite (for S^{m-1}), for any $\Theta \subset S^{m-1}$ of cardinality n, for some fixed n.

(b) Positive definite (for S^{m-1}), for any $\Theta \subset S^{m-1}$.

(c,d) Same as (a,b), with "invertibility" replacing "positive definiteness".

Definition 1.4. *Let g be a univariate continuous function defined on $[0, \pi]$. We say that g is* **(strictly) positive definite of order** n **for** S^{m-1} *if for each $\Theta \subset S^{m-1}$ of cardinality n the corresponding matrix A_Θ is (strictly) positive definite. A function g, that is (strictly) positive definite of all orders, is* **(strictly) positive definite.**

The analogous problem in Euclidean spaces, i.e., when $\Theta \subset \mathbb{R}^m$, has been intensively studied in the literature. In [S1], Schoenberg had characterized the positive definite functions of all orders for $\mathbb{R}^m$, and in [M], Micchelli established the invertibility of certain interpolation matrices arising from approximating scattered data in $\mathbb{R}^m$. Micchelli's results had led to a wealth of results (cf. e.g. [NW1,2], [QSW], [SiWa]), in which estimates for various norms and corresponding condition numbers of the interpolation matrix A were established. We refer to the review article of Dyn [D] for more information on this subject. Schoenberg also considered the problem on the sphere. In [S2] he proved the following characterization for a function g to be positive definite of all orders. In this result, $P_k^{(\lambda)}$ denotes the kth degree Gegenbauer ("ultraspherical") polynomial associated with λ, [Sz, p.81], [SW, p.148].

Result 1.5 [S2]. *A continuous function $g : [0, \pi]$ is positive definite on S^{m-1} if and only if it has the form*

$$g(t) = \sum_{k=0}^{\infty} a_k P_k^{(\lambda)}(\cos t), \tag{1.6}$$

in which $\lambda = (m-2)/2, a_k \geq 0$, and $\sum a_k P_k^{(\lambda)}(1) < \infty$.

Our interest in the problem was initiated by the paper [XC] of Xu and Cheney, in which the following question is addressed: "find conditions on the coefficients $(a_k)_{k \in \mathbb{N}}$ in (1.6), under which g is strictly positive definite" (either of a specific order or of all orders). Among various other results, it is shown in [XC] that, if all the coefficients a_k in (1.6) are positive, then the function g is strictly positive definite on S^{m-1}. Further discussions of this problem can be found in [LC] and [CX].

As this paper will show, a close relationship exists between the problem of determining strict positive definiteness and that of multivariate polynomial interpolation. This connection is discussed in §2, and allows us to find an equivalent version to the original problem in terms of polynomial interpolation on the unit sphere.

We then present three different methods for analysing the equivalent polynomial interpolation problem. The first exploits the de Boor-Ron "least solution for the polynomial interpolation problem" (cf. [BR1-3]). This method yields that g is strictly positive definite of order n if (but not only if) the corresponding coefficients $a_0, a_1, \ldots, a_{\lfloor n/2 \rfloor}$ are positive (cf. Theorem 3.2. The proof, as well as a complementary discussion, are provided in §4). For the case $m = 2$, this result was already proved in [XC]. Another approach takes advantage of the connection between spherical harmonics on the circle and analytic polynomials, and allows us to characterize, for $m = 2$, the strict positive definiteness of g in terms of the positive integer zeros of univariate exponentials with real coefficients. Our principal findings that results from this approach are collected in Theorem 3.5, with proofs and further discussion to be found in §5. Finally, in §6, we choose a different track which is suitable for any spatial dimension, and makes use of the fact that the harmonic polynomials are the polynomials in the kernel of the laplacian. That direction utilizes some of the polynomial ideal basics, and its main results are collected in Theorems 3.6 and 3.7. These theorems imply, in particular, that g is strictly positive definite if the set $K := \{k \in \mathbb{Z}_+ : \ a_k > 0\}$ (with (a_k) as in 1.6) contains arbitrarily long sequences of consecutive even integers, as well as arbitrarily long sequences of consecutive odd integers.

We use in the paper the following notations and conventions.

$$\Pi$$

stands for the space of all polynomials in m variables (where the value of m should be clear from the context). The subspace of Π that consists of the k-degree homogeneous polynomials is denoted by

$$\Pi_k^0.$$

Also, given any $K \subset \mathbb{Z}_+$, we set

$$\Pi_K := \sum_{k \in K} \Pi_k^0.$$

If $K = \{0, 1, 2, \ldots, k\}$, we often use $\Pi_k := \Pi_K$, i.e., Π_k is the space of all polynomials of degree $\leq k$. A parallel set of notations is used for *harmonic* polynomials. Here, we set

$$\mathcal{H}$$

for the space of all harmonic polynomials, and define

$$\mathcal{H}_k^0 := \Pi_k^0 \cap \mathcal{H}, \quad \mathcal{H}_K := \Pi_K \cap \mathcal{H}, \quad \mathcal{H}_k := \Pi_k \cap \mathcal{H}.$$

Finally, an **exponential** in this paper is either a function of the form

$$e_\theta : \mathbb{R}^m \to \mathbf{C} : x \mapsto e^{\theta \cdot x}, \quad \theta \in \mathbf{C}^m, \tag{1.7}$$

or any finite linear combination of such functions.

Additional definitions are used in the material presented in §6, and, to a lesser extent, in the last part of §4. Most of these are given in §3 (prior to the statement of Theorem 3.6, which is taken from §6).

2. The connection to polynomial interpolation by harmonic polynomials

Since we are seeking conditions that characterize the strict positive definiteness of g, and since Schoenberg's theorem already characterizes the positive definiteness of g, we may assume that g has the form (1.6). In turn, that allows us to write the matrix A_Θ as the infinite sum

$$A_\Theta = \sum_{k=0}^{\infty} a_k A_k, \tag{2.1}$$

with

$$A_k := (P_k^{(\lambda)}(\theta \cdot \vartheta))_{\theta,\vartheta\in\Theta}.$$

Since each summand A_k is positive definite (by virtue of Schoenberg's result, but also in view of (2.3) below), this immediately shows that the matrix A is more likely to be strictly positive definite, with the increase of the non-zero coefficients $(a_k)_k$ in the representation (1.6) of g. Also, since the matrix A_k is positive definite, it can be written in the form $C_k^T C_k$. Among such factorizations of A_k, we select below a particular one, which is based basic properties of *spherical harmonics*, and which will allow us to draw the connection between Problem 1.3 and polynomial interpolation on the sphere.

Recall that the restriction to S^{m-1} of a homogeneous harmonic polynomial of degree k is called a **spherical harmonic** of degree k. The following fact about spherical harmonics can be found, for example, in Stein and Weiss [SW, Chapter IV].

Let $\{Y_1^{(k)}, \ldots, Y_{h_k}^{(k)}\}$ be an orthonormal basis of $\mathcal{H}_k^0$ (here, $h_k := \dim \mathcal{H}_k^0$). Then, there is a positive constant $c_{k,\lambda}$ such that

$$P_k^{(\lambda)}(x \cdot y) = c_{k,\lambda} \sum_{j=1}^{h_k} Y_j^{(k)}(x) Y_j^{(k)}(y). \tag{2.2}$$

Here $\lambda = (m-2)/2$, and, as before, $P_k^{(\lambda)}$ is the appropriate Gegenbauer polynomial.

Using (2.2), we observe that A_k of (2.1) can be factored as follows:

$$A_k = c_{k,\lambda} B_k^T B_k, \tag{2.3}$$

with

$$B_k := B_{k,\Theta} := (Y_\mu^{(k)}(\theta))_{\mu=1,\,\theta\subset n\Theta}^{h_k}. \tag{2.4}$$

Therefore, for any vector $c \in \mathbb{R}^\Theta$, $c^T A_k c = \|B_k c\|_2^2$, and hence

$$c^T Ac = \sum_{k=0}^{\infty} a_k \|B_k c\|_2^2. \tag{2.5}$$

This proves one of the implications in Schoenberg's theorem, and implies that the problem of whether g is *strictly* positive definite or not depends only on the set

$$K_{m,g} := \{k \in \mathbb{Z}_+ :\ a_k \neq 0\}. \tag{2.6}$$

and not on the actual (positive) value of each a_k, $k \in K_{m,g}$.

Proposition 2.7. *Let g_1 and g_2 be two positive definite functions on S^{m-1}. If $K_{m,g_1} = K_{m,g_2}$, then, for every $\Theta \subset S^{m-1}$, the matrix $A_{g_1,\Theta}$ is invertible if and only if the matrix $A_{g_2,\Theta}$ is invertible.*

Because of the above proposition, we modify our Problem 1.3 as follows:

Modified Problem 2.8. *Study the following notion: Given a subset $K \subset \mathbb{Z}_+$, we say that K* **induces strict positive definiteness (of order** *n***)** *for S^{m-1} if every/some positive definite g that satisfies $K_{m,g} = K$ is strictly positive definite (of order n).*

Remark 2.9. Note that whenever g is strictly positive definite for S^{m-1}, it is certainly so for S^{l-1}, $l < m$. In view of Schoenberg's result, it can therefore be expressed as a linear combination of Gegenbauer polynomials of smaller orders. In this regard, it is useful to recall the following connection between Gegenbauer polynomials of different orders (cf. [AF]): for every $\lambda > \nu > 0$, and a non-negative k, there exist *positive* coefficients $(a_{\lambda,\nu,k,n})_n$ such that

$$P_k^{(\lambda)} = \sum_{0\le 2n\le k} a_{\lambda,\nu,k,n} P_{k-2n}^{(\nu)}.$$

As stated before, we begin our study of Problem 2.8 by linking it to the problem of interpolating by spherical harmonics on the sphere.

Theorem 2.10. *Given $K \subset \mathbb{Z}_+$, and a positive definite g that satisfies $K_{m,g} = K$, the following conditions are equivalent, for any $\Theta \subset S^{m-1}$.*

(a) The matrix $A_{g,\Theta}$ is invertible.

(b) The restriction of $\mathcal{H}_K$ to Θ is of full dimension $\#\Theta$, i.e., every $f : \Theta$ can be interpolated by a polynomial $p \in \mathcal{H}_K$.

Proof. By (2.5), $A_{g,\Theta}$ is singular if and only if $B_k c = 0$, for some $c \in \mathbb{R}^{\Theta}\backslash 0$, and for all $k \in K$. This is equivalent to the saying that the linear functional $\phi := \sum_{\theta\in\Theta} c_\theta \delta_\theta$ (with δ_θ the evaluation at θ) vanishes on a basis for each $\mathcal{H}_k^0$, $k \in K$, hence vanishes on $\mathcal{H}_K$. Finally, the existence of a non-zero linear functional that is supported on Θ and annihilates $\mathcal{H}_K$ is equivalent to the inequality $\dim(\mathcal{H}_{K|_\Theta}) < \#\Theta$. ♠

3. Main Results

Lemma 3.1 is proved in §4. Theorem 3.5 follows from Theorem 5.1, Corollary 5.2, Theorem 5.3, and Corollary 5.7 of §5. Theorems 3.6 and 3.7 are proved in §6. No formal proof is required for the other results stated in this section.

The space Π_Θ mentioned in following statement is the least solution of the polynomial interpolation problem, [BR1].

Lemma 3.1. *For any $\Theta \subset S^{m-1}$ of cardinality n, the following is true:*

(a) There exists a space $\Pi_\Theta \subset \mathcal{H}_{\lfloor n/2\rfloor}$ such that interpolation from Π_Θ to any f defined on Θ is always possible, i.e., the restriction of Π_Θ to Θ is of dimension $n = \#\Theta$.

(b) If n is even, and Θ does not lie on a circle, then $\Pi_\Theta \subset \mathcal{H}_{n/2-1} = \mathcal{H}_{\lfloor n/2\rfloor-1}$.

(c) If $n \neq 5$ is odd and no $(n-1)$ points of Θ lie on a circle, then $\Pi_\Theta \subset \mathcal{H}_{(n-3)/2} = \mathcal{H}_{\lfloor n/2\rfloor-1}$.

Combining Theorem 2.10 with (a) of Lemma 3.1, we arrive at the following result.

Theorem 3.2. *Let $K \subset \mathbb{Z}_+$. Then, given any $m \geq 2$, K induces strict positive definiteness of order n for S^{m-1} if $\{0, 1, \ldots, \lfloor n/2\rfloor\} \subset K$.*

For $m = 2$, the above theorem is due to [XC]. Note that the theorem reproduces the Xu-Cheney result which asserts that g is strictly definite positive whenever $K_{m,g} = \mathbb{Z}_+$. The statement of the theorem is sharp in the sense that the set $K := \{0, 1, \ldots, \lfloor n/2\rfloor - 1\}$ does *not* induce strict positive definiteness of order n. However, parts (b) and (c) in Lemma 3.1 imply that, for $m > 2$, this latter K "almost" induces the required strict positive definiteness.

Corollary 3.3. *Let g be a positive definite function for S^{m-1}, $K := K_{m,g}$. Assume that $\{0, 1, \ldots, \lfloor n/2\rfloor - 1\} \subset K$, and let $\Theta \subset S^{m-1}$ be of cardinality n. Then the corresponding matrix A_Θ of (1.1) is strictly positive definite if one of the following conditions holds:*

(a) n is even, Θ does not lie on a circle.

(b) n is odd, and no $n-1$ points from Θ lie on a circle.

Applying Remark 2.9 to Theorem 3.2, we arrive at the following corollary. A different proof for that corollary, that follows from the results of §6, is sketched at the end of the present section.

Corollary 3.4. *If g is positive definite for S^{m-1}, and if $K_{m,g}$ contains one even integer k_1, as well as one odd integer k_2, then g is strictly positive definite of order n for S^{l-1}, provided that $l < m$, and $\lfloor n/2 \rfloor \le \min(k_1, k_2)$.*

Theorem 3.2 is useful if one wants to assert the strict positive definiteness of g of a certain order n, since it allows g to be a polynomial of small degree. However, the theorem provides no real clue to the problem of strict positive definiteness (of all orders) of g, since it only implies (the aforementioned result) that g is strictly positive definite whenever $K_{m,g} = \mathbb{Z}_+$. Results concerning the induction of strict positive definiteness of all orders are derived in §5 and §6.

Section 5 is devoted to the case $m = 2$. Some of its findings are collected in our next theorem.

Theorem 3.5. *Let $K \subset \mathbb{Z}_+$. Then,*

(a) K induces strict positive definiteness for S^1 if and only if there exists no non-zero function of the form

$$\sum_{\tau \in \mathrm{T}} c_\tau e_{i\tau}, \quad \mathrm{T} \subset [0, 2\pi), \ \#\mathrm{T} < \infty, \quad c_\tau \in \mathbb{R}, \ \forall \tau$$

that vanishes on K.

(b) In order that $K \subset \mathbb{Z}_+$ induce strict positive definiteness for S^1, it is necessary that K has infinite intersection with any set of the form $k\mathbb{Z}_+$, $k \in \mathbb{N}$. The same applies to sets of the form $k/2 + k\mathbb{Z}_+$ (provided that k is even). However, K need not intersect at all sets of the form $\alpha + k\mathbb{Z}_+$, if $\alpha \neq 0, k/2$. In fact, for such α, $K := \mathbb{Z}_+ \backslash (\alpha + k\mathbb{Z}_+)$ induces strict positive definiteness for S^1.

(c) In order that $K \subset \mathbb{Z}_+$ induce strict positive for S^1, it suffices that, for every finite set J in $\mathbb{N}\backslash\{1\}$, and any $n \in \mathbb{N}$ there exist $\alpha \in \mathbb{Z}_+$ and $k \in \mathbb{N}\backslash(J\mathbb{N})$, such that $\{\alpha, \alpha + k, \ldots, \alpha + (n-1)k\} \subset K$.

In §6, we investigate the case of general m. There, we use the following additional notations and terminology.

The first required notion is that of **exponential spaces**. These are defined in terms of some set $\Omega \subset \mathbb{R}^m$, and a positive integer n as follows:

$$E_n(\Omega) := \{\sum_{\theta \in \Theta} c_\theta e_\theta : \ \Theta \subset \Omega, \ \#\Theta \le n, \ c_\theta \in \mathbb{C}, \forall \theta\}.$$

(Note that c_θ is a constant, but e_θ is a function.) Obviously, the above exponential spaces are not linear spaces, in contrast with the larger space

$$E(\Omega) := \cup_{n\geq 1} E_n(\Omega).$$

Another set of notations concerns maps defined on the algebra $\mathbf{A}$ of all formal power series in m-variables. Since $\mathbf{A}$ is the direct (infinite) sum $\sum_{k\geq 0} \Pi_k^0$ of the homogeneous polynomial spaces, there exists, for every $k \geq 0$, a well-defined projector

$$\overline{k} : \mathbf{A} \to \Pi_k^0,$$

that assigns to each power series $f \in \mathbf{A}$ the k-degree homogeneous component in its power expansion. Further, given a subset $K \subset \mathbb{Z}_+$, the sum

$$\overline{K} := \sum_{k\in K} \overline{k}$$

defines an analogous projector, this time from $\mathbf{A}$ onto Π_K.

Finally, we reserve a special notation Γ for the polynomial

$$\Gamma(x) := \Gamma_m(x) := \sum_{n=1}^{m} x_n^2.$$

Thus, $\Gamma\mathbf{A}$ is the (homogeneous principal) ideal generated by Γ. Note that the evaluation $\Gamma(D)$ of Γ at D is the laplacian operator.

The principal result of §6 is as follows:

Theorem 3.6. *Let $K \subset \mathbb{Z}_+$, and let n be a positive integer. Then K does not induce strict positive definiteness of order n for S^{m-1}, if and only if there exists $f \in E_n(S^{m-1})$ for which $\overline{K}(f) \in \Gamma\mathbf{A}$.*

In fact, the proof of Theorem 3.6 is more informative than its statement: given $\Theta \subset S^{m-1}$, the proof shows that the matrix $A_{g,\Theta}$ is singular if and only if there exists an exponential $f \in E(\Theta)$ that satisfies $\overline{K_{m,g}}(f) \in \Gamma\mathbf{A}$.

The following sufficient condition is derived in §6 from the characterization in Theorem 3.6.

Theorem 3.7. *Given $\Theta \subset S^{m-1}$, define*

$$\sigma(\Theta) := \min\{\#\Theta' \subset \Theta : \mathrm{span}(\Theta\backslash\Theta') \neq \mathbb{R}^m\}.$$

Let j be the minimal integer that satisfies $\binom{j+m-2}{m-1} > \sigma(\Theta)$. Let g be a positive definite function for S^{m-1}, and assume that the set $\{k \in K_{m,g} : \ k \geq (\#\Theta)/2\}$ contains j consecutive even integers as well as j consecutive odd integers. Then $A_{g,\Theta}$ is invertible.

Since $\sigma(\Theta) \leq n - m + 1$ for any $\Theta \subset \mathbb{R}^m$ of cardinality n, we obtain the following corollary:

Corollary 3.8. *Let $K \subset \mathbb{Z}_+$, and n a positive integer. Then K induces strict positive definiteness of order n on S^{m-1}, if, with j the minimal integer that satisfies $\binom{j+m-2}{m-1} > n-m+1$, there are j consecutive even integers and j consecutive odd integers in the set $\{k \geq n/2\} \cap K$.*

In particular, if K contains arbitrarily long sequences of consecutive evens and of consecutive odds, then K induces strict positive definiteness (of all orders) on S^{m-1}, for every $m \geq 2$.

Theorem 3.7 also provides another proof for Corollary 3.4: if Θ lies on S^{l-1}, $l < m$, then, in particular, it lies on linear proper subspace of $\mathbb{R}^m$. This implies that $\sigma(\Theta) = 0$, and one can take in Theorem 3.7 $j = 1$.

4. The least solution of the polynomial interpolation problem

We prove here Lemma 3.1.

Given any polynomial of degree k, we denote by $p_\uparrow$ the unique polynomial in Π^0_k that satisfies $\deg(p - p_\uparrow) < k$, and refer to $p_\uparrow$ as **the leading term** of p. Also, given $p \in \Pi$, we use $p(D)$ to denote the corresponding constant coefficient differential operator. Directional derivatives are denoted by D_x, where $x \in \mathbb{R}^m$ is the direction.

Given any finite $\Theta \subset \mathbb{R}^m$, the paper [BR1] introduces a polynomial space Π_Θ that satisfies all the following properties:

(a) For any function $f : \Theta$, there exists a unique $p \in \Pi_\Theta$ that agrees with f on Θ.

(b) Π_Θ is D-invariant, i.e., for every $x \in \mathbb{R}^m \backslash 0$, $D_x \Pi_\Theta \subset \Pi_\Theta$ (equivalently, Π_Θ is translation-invariant).

(c) Π_Θ is *homogeneous*. That is

$$\Pi_\Theta = \sum_{j=0}^{\infty} \Pi_{\Theta,j},$$

where $\Pi_{\Theta,j} := \Pi_\Theta \cap \Pi^0_j$.

(d) If the polynomial p vanishes on Θ, then its leading term $p_\uparrow$ annihilates Π_Θ in the sense that $p_\uparrow(D)\Pi_\Theta = 0$.

(e) Conversely, every homogeneous polynomial that annihilates Π_Θ is the leading term of some other polynomial that vanishes on Θ.

We refer to [BR1] and [BR3] for more details about Π_Θ. The exact definition of Π_Θ will be given in the sequel, and will be used for the proof of (c) in Lemma 3.1. For the time being, though, we need only the fact that a polynomial space that satisfies these five properties exists.

Lemma 4.1. *Let k be the least integer that satisfies $\Pi_\Theta \subset \Pi_k$. If $\Theta \subset S^{m-1}$, then*

$$\dim \Pi_{\Theta,j} \geq 2. \quad j = 1, 2, \ldots, k-1.$$

Claim (a) of Lemma 3.1 follows from Lemma 4.1. Indeed, since Π_Θ is translation-invariant, it must contain the constants. hence $\dim \Pi_{\Theta,0} = 1$. Also. by the definition of k, $\dim \Pi_{\Theta,k} \geq 1$. Hence, assuming Lemma 4.1 has been proved, we obtain that

$$\#\Theta = \dim \Pi_\Theta \geq 1 + 2(k-1) + 1 = 2k,$$

with the right-most equality due to property (a) of Π_Θ.

Proof of Lemma 2.2. We assume that the claim of the lemma is false. and will seek a contradiction. Let $0 < j < k$ be the maximal integer that violates the lemma's claim. If $j = k-1$, then $\Pi_{\Theta,j+1} = \Pi_{\Theta,k} \neq 0$, by the definition of k. Otherwise. by the maximality of j, $\dim \Pi_{\Theta,j+1} \geq 2$. Either way, $\Pi_{\Theta,j+1}$ contains a non-zero polynomial p. We consider the map $\psi : \mathbb{R}^m \to \Pi_j^0$ defined by

$$\psi(x) = D_x p.$$

Since Π_Θ is D-invariant, $\operatorname{ran} \psi \subset \Pi_\Theta$, hence $\operatorname{ran} \psi \subset \Pi_{\Theta,j}$. Since $\deg p > 0$, $\psi \neq 0$. On the other hand, $\dim \operatorname{ran} \psi \leq \dim \Pi_{\Theta,j} \leq 1$. Therefore, $\dim \operatorname{ran} \psi = 1$, hence $\dim \ker \psi = m-1$. Since ψ is linear, its kernel, then, is a hyperplane, and p has to be a univariate polynomial of the form

$$p(y) = c(\xi \cdot y)^{j+1},$$

with ξ perpendicular to the above hyperplane.

Let $\Gamma(D)$ be the m-dimensional Laplacian. We note that, since $j \geq 1$, $\Gamma(D)p = c' \sum_{i=1}^m \xi_i^2 (\xi \cdot y)^{j-1} \neq 0$. On the other hand, the quadratic polynomial $q := \Gamma - 1$ vanishes on Θ, hence by property (d) of Π_Θ, $\Gamma(D) = q_\uparrow(D)$ annihilates the entire Π_Θ. In particular, $\Gamma(D)p = 0$, which contradicts the previous conclusion. ♠

The proof of (b) in Lemma 3.1 is also quite simple. We present, however, a slightly longer proof, which prepares also for the proof of (c). We assume that n is even, and that $\Pi_\Theta \not\subset \Pi_{n/2-1}$, and will prove that Θ lies on a circle. First, by (a) of Lemma 3.1, $\Pi_\Theta \subset \Pi_{n/2}$. Second, we have $\dim \Pi_{\Theta,0} = 1$, and, since we assume that $\dim \Pi_{\Theta,n/2} \geq 1$, Lemma 4.1 implies that $\dim \Pi_{\Theta,j} \geq 2$, $j = 1, 2, \ldots, n/2 - 1$. Taking into account the fact that Π_Θ is n-dimensional, we realize that none of these inequalities can be sharp, i.e., the homogeneous dimensions of Π_Θ are

$$1, 2, 2, \ldots, 2, 1.$$

Let $q \in \Pi_{\Theta,j}\backslash 0$, $1 < j \leq n/2$, and let ψ_q be the corresponding map that was introduced in the proof of Lemma 4.1. The proof of that lemma shows that $\operatorname{ran}\psi_q \subset \Pi_{\Theta,j-1}$. Since $\dim \Pi_{\Theta,j-1} = 2$, this implies that $\dim\ker\psi_q \geq m-2$, i.e., that q is a bivariate polynomial. Further, the argument in Lemma 4.1 makes clear that q cannot be univariate. Consequently, $\operatorname{rank}\psi_q = 2$, and $\operatorname{ran}\psi_q = \Pi_{\Theta,j-1}$. By selecting p to be any non-zero polynomial in $\Pi_{\Theta,n/2}$, and choosing $\alpha \in \mathbb{Z}_+^d$ such that $|\alpha| = n/2-j$ and $q := D^\alpha p \neq 0$, we obtain that each $\Pi_{\Theta,j-1}$, $j = 2,\ldots,n/2$, is generated by derivatives of p. That same result is trivial for $j = n/2+1$ (since $\Pi_{\Theta,n/2}$ is 1-dimensional), and for $j = 1$. Thus,

$$\Pi_\Theta = \{P(D)p : P \in \Pi\},$$

i.e., Π_Θ comprises of the derivatives of the single polynomial p, hence, in particular, Π_Θ is annihilated by the $(m-2)$-dimensional space $\{D_x : x \in \ker\psi_p\}$. Selecting any basis for that space, we invoke property (e) of Π_Θ to conclude that there exist $m-2$ linearly independent linear polynomials, each of which vanishes on Θ, i.e., that Θ lies on a 2-dimensional linear manifold. Since Θ is assumed to lie also on S^{m-1}, we conclude that, indeed, it lies on a circle. This completes the proof of (b).

Before we prove (c), we mention that its statement is sharp in the following sense. First, if Θ consists of five points in S^2, then, regardless of its distribution, the 5-dimensional Π_Θ cannot lie in the 4-dimensional $\Pi_{(n-3)/2} = \Pi_1$, i.e., the statement in (c) fails to hold for $n = 5$. Further, if all the points of Θ except one lie on a circle, then $\Pi_\Theta \not\subset \Pi_{(n-3)/2}$, and, therefore, one cannot prove in (c) that Θ must entirely lie on a circle.

In order to prove (c), we assume that $n \geq 7$ is odd, and that $\Pi_\Theta \not\subset \Pi_{(n-3)/2}$ (the claim is trivial for $n = 1, 3$). We need to prove then that, save perhaps one point, Θ lies on a circle. Here, we need to recall from [BR1] that the definition of Π_Θ is

$$\Pi_\Theta = \operatorname{span}\{f_\downarrow : f \in E(\Theta)\}, \tag{4.2}$$

with $E(\Theta)$ (as before) the span of the exponentials e_θ, $\theta \in \Theta$, and with $f_\downarrow$ the smallest degree non-zero homogeneous term is the power expansion of f, (i.e., with k the minimal non-negative intege · 'hat satisfies $\overline{k}(f) \neq 0$, $f_\downarrow := \overline{k}(f)$). It is then easy to check that, if Θ contains four points which are not co-planar, then $\dim \Pi_{\Theta,1} \geq 3$. Therefore, assuming Θ not to lie on a circle, we must have that $\dim \Pi_{\Theta,1} \geq 3$. Repeating the same counting arguments that we employed in the proof of (b), we conclude that the homogeneous dimensions of Π_Θ must be

$$1, 3, 2, 2, \ldots, 2, 1.$$

Selecting any $p \in \Pi_{\Theta,(n-1)/2}$, and repeating the argument that was used in the proof of (b), we conclude that p is bivariate and that the derivatives of p form a subspace in Π_Θ of dimension $n-1$ (the argument relies on the fact that $\dim \Pi_{\Theta,(n-1)/2-1} = 2$, hence requires

$n \geq 7$). Let $x \in \mathbb{R}^m$ be such that $D_x p = 0$. Let also $f \in E(\Theta)$ be an exponential that satisfies $f_\downarrow = p$. Set $g := D_x f$, $k := (n-1)/2$. Since $\overline{j}(f) = 0$, for $j < k$, we have that $(\overline{j-1})(g) = 0$. Also, since $\overline{k}(f) = p$, we have $(\overline{k-1})(g) = (\overline{k-1})(D_x f) = D_x \overline{k}(f) = D_x p = 0$. Therefore, $\deg(g_\downarrow) \geq k$. Since $g_\downarrow$ is in Π_Θ, and $\Pi_\Theta \subset \Pi_k$, we must thus have $\deg(g_\downarrow) = k$, hence that $g_\downarrow$ is a constant multiple of p (since $\dim \Pi_{\Theta,k} = 1$), say, $g_\downarrow = cp$. This implies that, with q the linear polynomial $q(y) := x \cdot y - c$, $\overline{j}(q(D)f) = 0$, $0 \leq j \leq k$, and thereby that, if $q(D)f \neq 0$, then $\deg(q(D)f)_\downarrow > k$, in contradiction with the fact that $\Pi_\Theta \subset \Pi_k$. Thus, $q(D)f = 0$.

On the other hand, as any function in $E(\Theta)$, f can be written in the form $f = \sum_{\theta \in \Theta} c_\theta e_\theta$, hence $0 = q(D)f = \sum_{\theta \in \Theta} q(\theta) c_\theta e_\theta$. Since any finite set of exponentials are linearly independent, we conclude that

$$c_\theta \neq 0 \quad \Longrightarrow \quad q(\theta) = 0.$$

Thus, the subset

$$\Theta' := \{\theta \in \Theta : \ c_\theta \neq 0\}$$

lies in the hyperplane $q = 0$. Ranging the directional derivative D_x over an $(m-2)$-dimensional space (which is possible since p is bivariate) we obtain, as in the proof of (b), that Θ' lies on a circle. It remains to show that $\#\Theta' \geq n - 1$. However, since $f \in E(\Theta')$, and $p = f_\downarrow$, $p \in \Pi_{\Theta'}$, and therefore its space of derivatives $D(p)$ lies in $\Pi_{\Theta'}$, as well. Since $\dim D(p)$ was shown to be $n-1$, we conclude that $\#\Theta' = \dim \Pi_{\Theta'} \geq \dim D(p) = n - 1$.

5. Strict positive definiteness of K for S^1: an analytic approach

In the case of interpolating on the circle, $\dim \mathcal{H}_k^0 = 2$ for all $k = 1, 2, \ldots$, and $\mathcal{H}_k^0$ is spanned by the two functions $\cos k\tau$ and $\sin k\tau$, where (r, τ) are the polar coordinates in $\mathbb{R}^2$. These well-known facts can be nicely used in the course of study of Problem 2.8. We will connect our problem to the distribution of zeros of bounded univariate exponentials, and use the obtained characterization to derive separate necessary and sufficient conditions for the induction of strict positive definiteness by K for S^1.

Throughout the section, we will make an essential use of the following univariate exponential space:

$$\mathcal{E}_n := \{ \sum_{\tau \in \mathrm{T} \subset [0,2\pi)} c_\tau e_{i\tau} : \ \#\mathrm{T} \leq n, \ \ c_\tau \in \mathbb{R}, \ \forall \tau \}.$$

Also, $\mathcal{E} := \cup_{n \geq 1} \mathcal{E}_n$. Note that, importantly, we allow only *real* coefficients c_τ in the definition.

We first record and prove the basic observation that will be utilized throughout this section.

Theorem 5.1. *Given $K \subset \mathbb{Z}_+$, the following conditions are equivalent.*
(a) K induces strict positive definiteness for S^1 of order n.
(b) K is a uniqueness set for $\mathcal{E}_n$, i.e., only the 0 function in $\mathcal{E}_n$ vanishes identically on K.

Proof. Selecting the basis

$$Y_1^{(k)}(\cos\tau, \sin\tau) := \cos k\tau, \quad Y_2^{(k)}(\cos\tau, \sin\tau) := \sin k\tau$$

for $\mathcal{H}_k^0$, and writing a given finite $\Theta \subset S^1$ in the form $\Theta = \{(\cos\tau, \sin\tau) : \tau \in \mathrm{T} \subset [0, 2\pi)\}$, a typical column of the matrix B_k of (2.4) then has the form

$$(\cos k\tau, \sin k\tau)^T.$$

Therefore, given $c \in \mathbb{R}^{\mathrm{T}}$, we conclude that

$$\|B_k c\|_2^2 = \Big(\sum_{\tau\in\mathrm{T}} c_\tau \cos k\tau\Big)^2 + \Big(\sum_{\tau\in\mathrm{T}} c_\tau \sin k\tau\Big)^2 = \Big|\sum_{\tau\in\mathrm{T}} c_\tau e^{ik\tau}\Big|^2.$$

Introducing the exponential

$$f : x \mapsto \sum_{\tau\in\mathrm{T}} c_\tau e^{i\tau x},$$

we conclude that $\|B_k c\|_2^2 = |f(k)|^2$. Now, suppose that K induces strict positive definiteness of order n, and f is any exponential of above form in $\mathcal{E}_n$. With $(B_k)_k$ the corresponding matrices, we have, for some $k \in K$, $B_k c \neq 0$, and hence, at that k $f(k) \neq 0$. The converse implication is obtained by reversing the argument. ♠

The last theorem leads to the following necessary condition for the induction of strict positive definiteness by $K \subset \mathbb{Z}_+$.

Corollary 5.2. *In order that $K \subset \mathbb{Z}_+$ induce strictly positive definite for S^1, it is necessary that K has infinite intersection with any set of the form $k\mathbb{Z}_+$, $k \in \mathbb{N}$. The same applies to sets of the form $k/2 + k\mathbb{Z}_+$, provided that k is even.*

Proof. In order to unify the proof, we assume that the relevant arithmetic progression is of the form $Z := \alpha + k\mathbb{Z}_+$, $\alpha < k$. We will further assume that K has only finite intersection with Z, and will use that to construct a linear combination f^* of exponentials $(e_{i\tau})_{\tau\in[0,2\pi)}$ that vanishes on K. The crux in the proof is that, if $\alpha = 0$ or $\alpha = k/2$, the coefficients in the representation $f^* = \sum_\tau c_\tau e_{i\tau}$ are *real*. Thus, for these cases, $f^* \in \mathcal{E}$, and the desired result is then implied by Theorem 5.1.

Set

$$\mathrm{T} := \{(2\pi l)/k : \ l = 0, 1, \ldots, k-1\},$$

and define the univariate exponential

$$f := \sum_{\tau \in \mathrm{T}} e_{i\tau}(\cdot - \alpha).$$

Then $f(l) = 0$, for $0 \le l \le k-1$, $l \ne \alpha$, and since f is k-periodic, we conclude that f actually vanishes on $\mathbb{Z}_+ \backslash Z$.

By our assumption, the set $K \cap Z$ is finite. Let n denote its cardinality, and let $t_0, t_1, t_2, \ldots, t_{2n}$ be chosen in a way that (a) each set $t_l + \mathrm{T}$ is a subset of $[0, 2\pi)$, and (b) the sets $t_l + \mathrm{T}$, $l = 0, \ldots, 2n$ are pairwise disjoint. The restriction to $K \cap Z$ of the $2n+1$ exponentials $\{e_{it_l} f\}_{l=0}^{2n}$ must be linearly dependent over *the reals*, since the set $K \cap Z$ contains no more than n points. Let f^* be a non-trivial linear combination with real coefficients of these functions that vanishes on $K \cap Z$. Furthermore, since f vanishes on $\mathbb{Z}_+ \backslash Z$, so does every $e_{it_l} f$, and therefore f^* vanishes on that set, as well. This implies that f^* vanishes on K, too. Finally, since the spectra of the exponentials $\{e_{it_l} f\}_l$ are pairwise disjoint, f^* cannot be identically 0 (since any finitely many exponentials are linearly independent).

It remains to show that $f^* \in \mathcal{E}$, and this is the part where we need the assumption $\alpha = 0, k/2$. Indeed, for such choice of α, we observe that, the numbers $e_{i\tau}(-\alpha)$, $\tau \in \mathrm{T}$ are real, hence, in these cases, in $f \in \mathcal{E}$, implying that $e_{it_l} f \in \mathcal{E}$, too. Since $\mathcal{E}$ is a linear space over the reals, we conclude that f^* lies in that space, too.

Consequently, we have found a non-trivial exponential $f^* \in \mathcal{E}$, that vanishes on K.♠

Note that the exponential f^* that was used in the proof of the above result lies in $\mathcal{E}_{(2n+1)k}$. This raises the following question: assuming K to contain exactly n numbers from the set $\alpha + k\mathbb{Z}_+$, is it possible that K induces strict positive definiteness on S^1 of orders smaller, still closer, to $(2n+1)k$ (or, roughly speaking, is the construction of f^* in the last proof uses as few exponentials as possible)? Some answer to this question is provided, for the set $1 + 2\mathbb{Z}_+$, by the choice $K := \{0, \ldots, 2n\}$. Here, by Theorem 3.2, K induces strict positive definiteness of order $4n+1$, which, in view of Theorem 5.1, implies that no non-zero exponential in $\mathcal{E}_{4n+1}$ can vanish on K. On the other hand, K contains n numbers from the set $1 + 2\mathbb{Z}_+$. Thus, for this K, we have used $(2n+1)2 = 4n+2$ frequencies to construct f^*, while the present argument shows that we could not use $4n+1$ frequencies. In summary, at least for that example, the construction of f^* was "economical".

Our next result is a stronger, more quantitative version, of statement (c) in Theorem 3.5.

Theorem 5.3. *Let K be a subset of $\mathbb{Z}_+$ and let n be a positive integer. Assume that, for every $J \in \mathbb{N} \backslash \{1\}$ of cardinality $n-1$, there exist $\alpha \in \mathbb{Z}_+$ and $k \in \mathbb{N} \backslash (J\mathbb{N})$ such that $\{\alpha, \alpha + k, \ldots, \alpha + (n-1)k\} \subset K$. Then K induces strict positive definiteness of order n on S^1.*

The following lemma is required for the proof of the theorem.

Lemma 5.4. *Let* T *be any finite subset of* $(0, 2\pi)$. *Then the set of integer zeros of the function* $F := \prod_{\tau \in \mathrm{T}}(e_{i\tau} - 1)$ *is of the form* $J\mathbb{Z}$, *with* J *a subset of* $\mathbb{N}\backslash 1$ *of cardinality* $\le \#\mathrm{T}$.

Proof. Let $\tau \in \mathrm{T}$. The condition $e_{i\tau}(k) = 1$ is equivalent to $k \in \frac{2\pi}{\tau}\mathbb{Z}$. Since $\frac{2\pi}{\tau} > 1$, the group $\mathbb{Z} \cap \frac{2\pi}{\tau}\mathbb{Z}$ is a proper subgroup of $\mathbb{Z}$. Thus, the zero set of F is the union of at most $\#\mathrm{T}$ proper subgroups of $\mathbb{Z}_+$. ♠

Proof of Theorem 5.3. Assuming K to satisfy the assumptions made in the theorem, we will also assume that there exists an exponential $f \in \mathcal{E}_n$ that vanishes on K, and will show that the two assumptions contradict each other. Our claim would then follow from Theorem 5.1.

Being in $\mathcal{E}_n$, the exponential f has the form

$$f = \sum_{\tau \in \mathrm{T}} c_\tau e_{i\tau}, \quad \mathrm{T} \subset [0, 2\pi), \ c_\tau \in \mathbb{R}\backslash 0,$$

with $\#\mathrm{T} \le n$. Without loss, we may assume that $0 \in \mathrm{T}$, and that $c_0 = 1$ (otherwise, we divide f by $c_{\tau_0} e_{i\tau_0}$, with τ_0 the smallest number in T). By Lemma 5.4, the positive integer zeros of the function

$$F := \prod_{\tau \in \mathrm{T}\backslash 0} (e_{i\tau} - 1)$$

are of the form $J\mathbb{N}$, with $J \subset \mathbb{N}\backslash 1$ of cardinality $< n$. By our assumptions on K, there exist $\alpha \in \mathbb{Z}_+$, and $k \in \mathbb{N}\backslash(J\mathbb{N})$, such that $\{\alpha, \alpha + k, \ldots, \alpha + (n-1)k\} \subset K$. For that k, $F(k) \ne 0$, that is $1 \notin \{e_{i\tau}(k)\}_{\tau \in \mathrm{T}\backslash 0}$.

Let p be a polynomial

$$p : t \mapsto \sum_j a_j t^j$$

whose zero set is $\{e_{i\tau}(k) : \tau \in \mathrm{T}\backslash 0\}$. Then, $\deg p < n$, and $p(1) \ne 0$. Let $p(\nabla)$ be the induced difference operator

$$p(\nabla) : g \mapsto \sum_j a_j g(\cdot + jk).$$

Then, $p(\nabla)e_{i\tau} = e_{i\tau}\, p(e_{i\tau}(k)) = 0$, for each $\tau \in \mathrm{T}\backslash 0$. Hence $p(\nabla)f = p(1) \ne 0$, i.e., $p(\nabla)f$ is a non-zero constant. However, since we assume that f vanishes at each of the points $\alpha + jk$, $j = 0, \ldots, n-1$, it follows that $p(\nabla)f$ must vanish at α. Thus, we have reached the desired contradiction. ♠

The value $k = 1$ is always admissible in Theorem 5.3. Hence:

Corollary 5.5. *If $K \subset \mathbb{Z}_+$ contains n consecutive integers, then K induces strict positive definiteness of order n on S^1.*

Another special case is the following:

Corollary 5.6. *Assume that $K = K\mathbb{Z}_+$. Then it induces strict positive definiteness on S^1, provided that it is not of the form $J\mathbb{Z}_+$, $\#J < \infty$.*

Proof. We invoke Theorem 5.3. For that, we let n be any positive integer, and J be any finite subset of $\mathbb{N}\backslash\{1\}$. By our assumption, there exist $k \in K\backslash(J\mathbb{N})$. The property $K = K\mathbb{Z}_+$ therefore implies that $\{k, 2k, \ldots, nk\} \subset K$. By virtue of Theorem 5.3, K induces strict positive definiteness of order n for S^1. Since n was arbitrary, our claim is proved. ♠

Finally, we complement Corollary 5.2 by the following result (which completes the proof of Theorem 3.5):

Corollary 5.7. *The set $K := \mathbb{Z}_+\backslash(\alpha + k\mathbb{Z}_+)$ induces strict positive definiteness for S^1, provided that $\alpha \notin \frac{k}{2}\mathbb{Z}_+$.*

Proof. Thanks to Theorem 5.1, it suffices to show that no exponential $f \in \mathcal{E}$ vanishes on K.

Let $f = \sum_{\tau\in\mathrm{T}} c_\tau e_{i\tau}$ be an exponential that vanishes on K. We will show that, necessarily, one of the coefficients (c_τ) is not a real number, implying thus that $f \notin \mathcal{E}$. Without loss (see the proof of Theorem 5.3) we can assume that $0 \in \mathrm{T}$, and that $c_0 = 1$. Further, we can also assume without loss that $\mathrm{T}_0 := \{0, 2\pi/k, \ldots, 2\pi(k-1)/k\} \subset \mathrm{T}$ (otherwise, we may add the missing exponentials with 0 coefficients). For $\tau \in \mathrm{T}_0$, and $t \in \mathrm{T}$, the difference operator $\nabla_t : g \mapsto g - e_{it}(-k)g(\cdot + k)$ annihilates e_{it}, and satisfies $\nabla_t(e_{i\tau}) = (1 - e_{it}(-k))e_{i\tau}$ (since $e_{i\tau}$ is k-periodic). Therefore, applying $\nabla := \prod_{t\in\mathrm{T}\backslash\mathrm{T}_0}\nabla_t$ to f, we obtain, with $c := \prod_{t\in\mathrm{T}\backslash\mathrm{T}_0}(1 - e_{it}(-k)) \neq 0$,

$$\nabla f = c \sum_{\tau\in\mathrm{T}_0} c_\tau e_{i\tau}.$$

On the other hand, it is clear that ∇f still vanishes on K, and in particular ∇f vanishes on $\{0, 1, \ldots, k-1\}\backslash\alpha$. But, the exponentials $\{e_{i\tau}\}_{\tau\in\mathrm{T}}$ are the characters of the group $\{0, \ldots, k-1\}$, hence they are linearly independent over that set, and consequently, since $c_0 = 1$, $c_\tau = e_{i\tau}(-\alpha)$, $\tau \in \mathrm{T}_0$. However, our assumption on α and k ensures that $\frac{2\pi\alpha}{k} \notin \pi\mathbb{Z}$, or, equivalently, that $c_{2\pi/k} = e_{2\pi i/k}(\alpha)$ is not a real number. ♠

6. Strict positive definiteness of K for S^{m-1}: an algebraic approach

Here, we attack the problem from a completely different angle. While the core of the argument used in the previous section was the connection between 2-dimensional harmonic polynomials and their analytic completions, the course here is of algebraic nature. It is based on the realization of harmonic polynomials as the kernel of the laplacian, thereby exploits the connection between a homogeneous polynomial ideal (here, the principal one generated by Γ) and its kernel in Π (here, the harmonic polynomials). We will require here the notations which were introduced before Theorem 3.6. Also, recall that a subspace H of $\mathbf{A}$ or Π is **homogeneous** if each of the maps $\overline{k}$, $k \in \mathbb{Z}_+$, maps H into itself.

Some of the results in this section can be developed in a more general (and in our opinion more natural) setup.

Definition 6.1. *Let Ω be a subset of $\mathbb{R}^m$, H a homogeneous subspace of Π, and n a positive integer. We say that H is* **correct of order** n **on** Ω *if for any $\Theta \subset \Omega$ of cardinality n, and any function F defined on Θ there exists a polynomial $p \in H$ that interpolates F (on Θ).*

To see the connection between the new definition and our original problem, one chooses $\Omega := S^{m-1}$, and $H := \mathcal{H}_K$ (defined as in the introduction). Then, as asserted in Theorem 2.10, the induction by K of strict positive definiteness of order n on S^{m-1} is equivalent to the nth-order correctness of H on S^{m-1}.

Theorem 6.2. *Let H, n and Ω be as in the above definition. Then H is correct of order n on Ω if and only if for every exponential $f \in E_n(\Omega)$, there exists $k \in \mathbb{Z}_+$ such that the kth order homogeneous differential operator $\overline{k}(f)(D)$ does not annihilate $H \cap \Pi_k^0$.*

Proof. For $\theta \in \mathbb{R}^m$, let δ_θ be the functional $\delta_\theta : f \mapsto f(\theta)$, and for a finite $\Theta \subset \mathbb{R}^m$, let δ_Θ be $\mathrm{span}\{\delta_\theta\}_{\theta\in\Theta}$. H fails to be correct of order n on Ω if and only if for some $\Theta \subset \Omega$ of cardinality n, $\dim H_{|\Theta} < \dim \delta_\Theta = n$. But in (and only in) such a case, there would be $\lambda = \sum_{\theta\in\Theta} c_\theta \delta_\theta \in \delta_\Theta$ which is orthogonal to H, i.e.,

$$\sum_{\theta\in\Theta} c_\theta p(\theta) = 0, \quad \forall \quad p \in H.$$

Defining $f := \sum_{\theta\in\Theta} c_\theta e_\theta$, we obtain an exponential $f \in E_n(\Omega)$ such that $p(D)f(0) = \lambda p = 0$, for every $p \in H$. For $p \in H \cap \Pi_k^0$, we have

$$0 = p(D)f(0) = p(D)(\overline{k}(f))(0) = (\overline{k}(f)(D))p(0) = (\overline{k}(f)(D))p.$$

Therefore, the condition $p(D)f(0) = 0$, all $p \in H$, is equivalent to $\overline{k}(f)(D)$ annihilating $H \cap \Pi_k^0$ for every k. ♠

Proof of Theorem 3.6. By Theorem 2.10 (and in view of Definitions 2.8 and 6.1) the required property of K is equivalent to $\mathcal{H}_K$ being correct of order n for S^{m-1}. Therefore, in view of Theorem 6.2, we need to prove that the condition stated in the present theorem is equivalent to the following: "There exists $f \in E_n(S^{m-1})$ such that, for every $k \in K$, $(\overline{k}(f))(D)$ annihilates $\mathcal{H}_K \cap \Pi_k^0 = \mathcal{H}_k^0$." Since $\mathcal{H}_k^0$ is the kernel in Π_k^0 of the laplacian $\Gamma(D)$, the last condition is equivalent to the divisibility of $\overline{k}(f)$ by Γ. ♠

In the rest of the section, we prove Theorem 3.7. We divide the proof into a sequence of several lemmas, some of them might appear to be of independent interest. Note that $\binom{j+m-2}{m-1} = \dim \Pi_{j-1}^0$.

Lemma 6.3. *The operator $\Gamma(D)$ induces an isomorphism between $\Gamma^{n+1}\Pi$ and $\Gamma^n\Pi$, $n = 0, 1, \ldots$.*

Proof. First, we recall that Π is the direct sum of the spaces $(\Gamma^k\mathcal{H})_{k\in\mathbb{Z}_+}$. (A quick proof of that would go as follows. Since $\mathcal{H}$ is the kernel in Π of the homogeneous differential operator $\Gamma(D)$, Π is the direct sum of $\mathcal{H}$ and the principal ideal $\Gamma\Pi$:

$$\Pi = \mathcal{H} \oplus \Gamma\Pi.$$

Multiplying the above equation by Γ^k, and using induction we obtain

$$\Pi = (\sum_{j=0}^{k-1} \Gamma^j\mathcal{H}) \oplus \Gamma^k\Pi = (\sum_{j=0}^{k} \Gamma^j\mathcal{H}) \oplus \Gamma^{k+1}\Pi.)$$

Next, one checks directly that, for each $k \in \mathbb{Z}_+$, there exists a constant $c_{n,k}$ such that the operator $c_{n,k}\Gamma\Gamma(D)$ is the identity on $\Gamma^{n+1}\mathcal{H}_k^0$. This implies that $\Gamma(D)$ maps $\Gamma^{n+1}\mathcal{H}$ 1-1 onto $\Gamma^n\mathcal{H}$.

Finally, the decomposition result asserted in the first paragraph of the proof allows us to write

$$\Gamma^{n+1}\Pi = \oplus_{k\geq n+1}\Gamma^k\mathcal{H}.$$

The desired result then follows from an application of $\Gamma(D)$ to both sides of the last equality, and invoking the isomorphism assertion from the second paragraph of the proof.

♠

Lemma 6.4. *Assume that $f \in \mathbf{A}$ and satisfies $\Gamma(D)f = f$. Assume further, that for some $k, n \in \mathbb{Z}_+$,*

$$(\overline{k+2j})(f) \in \Gamma\Pi, \quad j = 0, \ldots, n. \tag{6.5}$$

Then,

$$(6.6) \qquad \overline{(k+2n)}(f) \in \Gamma^{n+1}\Pi.$$

Proof. By induction on n. If $n = 0$, the claim in (6.6) is assumed in equation (6.5). Assume thus that the claim is valid for $n-1 \geq 0$. Since $\Gamma(D)f = f$, $\Gamma(D)(\overline{(k+2n)}(f)) = \overline{(k+2n-2)}(f)$, hence by the induction hypothesis

$$\Gamma(D)(\overline{(k+2n)}(f)) \in \Gamma^n\Pi.$$

Invoking Lemma 6.3, we conclude that

$$\overline{(k+2n)}(f) \in \mathcal{H} + \Gamma^{n+1}\Pi.$$

The result then follows from the assumption that $\overline{(k+2n)}(f)$ is divisible by Γ (and the fact that no polynomial in $\mathcal{H}\backslash 0$ is divisible by Γ). ♠

Proof of Theorem 3.7. Let $\Theta \subset S^{m-1}$ of cardinality n be given, and let $f \in E(\Theta)$. By Theorem 3.6 (more precisely, by the statement made in the paragraph that follows Theorem 3.6), we need to show that $\overline{K}(f)$ is not divisible by Γ. Assume, to the contrary, that $\overline{K}(f)$ *is* divisible by Γ. We first note that, for some $k \leq n/2$, say, k_0, $\overline{k_0}(f) \neq 0$. (Indeed, by the definition (4.2) of Π_Θ, if $\overline{k}(f) = 0$ for all $k \leq n/2$, then Π_Θ must contain a polynomial of degree $> n/2$, in contradiction with Lemma 3.1). Since f, as any function in $E(S^{m-1})$, satisfies $\Gamma(D)f = f$, one concludes from the fact that $\overline{k_0}(f) \neq 0$, that $\overline{i}(f) \neq 0$, for every $i \in k_0 + 2\mathbb{Z}_+$. Because $k_0 \leq n/2$, our assumptions on K imply that there exist j integers $k, k-2, k-4, \ldots, k-2j+2$ in K, that further satisfy $k - 2j + 2 - k_0 \in 2\mathbb{Z}_+$. Thus, the fact that $\overline{i}(f) \neq 0$ holds, in particular, for any integer i in this progression. On the other hand, each of these integers lies in K, and since $\overline{K}(f) \in \Gamma\Pi$, Lemma 6.4 implies that

$$\overline{k}(f) \in \Gamma^j\Pi.$$

As the rest of the proof will establish, this last conclusion contradicts the fact that $f \in E(\Theta)$.

Since f is a linear combination of $\{e_\theta\}_{\theta\in\Theta}$, we have that

$$\overline{k}(f) = \sum_{\theta\in T} c_\theta(\theta\cdot)^k,$$

with $T \subset \Theta$, and with any $\theta, \theta' \in T$ being linearly independent. Here,

$$(\theta\cdot)^k : x \mapsto (\theta\cdot x)^k.$$

Let T' be a non-spanning set in T of maximal cardinality. Since $\dim \Pi^0_{j-1} > \sigma(\Theta) \geq \#(T\backslash T')$, there exists a non-trivial homogeneous polynomial $p_1 \in \Pi^0_{j-1}$ that vanishes on $T\backslash T'$. Writing p_1 in the form $p_1 = \Gamma^i p$ with $p \notin \Gamma\Pi$ and $i \geq 0$, we conclude that p is a homogeneous polynomial of degree $r < j$ which vanishes on $T\backslash T'$. An application of $p(D)$ to $\overline{k}(f)$ yields a linear combination of the form

$$q = \sum_{\theta \in T'} a_\theta (\theta\cdot)^{k-r}.$$

Since T' does not span, the above polynomial is of $< m$ variables.

On the other hand, $\overline{k}(f) = \Gamma^s P$, for some $P \in \Pi\backslash(\Gamma\Pi)$, $s \geq j$. Therefore, for some $c \neq 0$,

$$p(D)(\overline{k}(f)) = p(D)(\Gamma^s P) = c\,p\Gamma^{s-r} P + \Gamma^{s-r+1} P_1.$$

Since cpP is non-zero and is not divisible by Γ, we conclude that $p(D)(\overline{k}(f))$ is a non-zero polynomial in $\Gamma\Pi$, hence, in particular, cannot be of less than m variables, and we thus have reached the desired contradiction. ♠

References

[AF] R. Askey and J. Fritch, Integral representations for Jacobi polynomials, and some applications, J. Math. Anal. Appl. 26 (1969), 411-437.

[BR1] C. de Boor and A. Ron, On multivariate polynomial interpolation, Constructive Approximation **6** (1990), 287–302.

[BR2] C. de Boor and A. Ron, Computational aspects of polynomial interpolation in several variables, Mathematics of Computations **58(198)** (1992), 705–727.

[BR3] C. de Boor and A. Ron, The least solution of the multivariate polynomial interpolation, Math. Z. **210** (1992), 347–378.

[CX] E.W. Cheney and Yuan Xu, A set of research problems in approximation theory, in *Topics in Polynomials of One and Several Variables and Their Applications* (Th. M. Rassias, H.M. Srivastava, and A. Yanushauskas, eds.), World Scientific, London (to appear).

[D] N. Dyn, Interpolation and approximation by radial and related functions, in *Approximation* VI (C. Chui, L. Schumaker, and J.D. Ward, eds.) (1989), Vol. I, 211-234.

[LC] W.A. Light and E.W. Cheney, Interpolation by periodic radial basis functions, J. Math. Anal. Appl. 168 (1992), 111-130.

[M] C. Micchelli, Interpolation of scattered data: distance matrices and conditionally positive definite functions, Constr. Approx. **2** (1986), 11-22.

[NW1] F.J. Narcowich and J.D. Ward, Norms of inverses and condition numbers of matrices associated with scattered data, J. Approx. Theory **64** (1991), 69-94.

[NW2] F.J. Narcowich and J.D. Ward, Norm estimates for the inverses of a general class of scattered data radial function interpolation matrices, J. Approx. Theory **69** (1992), 84-109.

[QSW] E. Quak, N. Sivakumar, and J.D. Ward, Least squares approximation by radial functions, ms 1991. SIAM J. Math. Anal., to appear

[S1] I.J. Schoenberg, Metric spaces and completely monotone functions, Ann. Math. **39** (1938), 811-841.

[S2] I.J. Schoenberg, Positive definite functions on spheres, Duke Math. J. **9** (1942), 96-108.

[SW] E.M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton, NJ, 1971.

[SiWa] N. Sivakumar and J.D. Ward, On the least squares fit by radial functions to multidimensional scattered data, Numer. Math. **65** (1993), 219–243.

[Sz] Gabor Szegö, *Orthogonal Polynomials*, Amer. Math. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, RI, 1959.

[XC] Yuan Xu and E.W. Cheney, Strictly positive definite functions on spheres, Proc. Amer. Math. Soc. 116 (1992), 977-981.